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Green's matrix for a second-order self-adjoint matrix differential operator

Tahsin Çağrı Şişman and Bayram Tekin

Department of Physics, Middle East Technical University, 06531, Ankara, Turkey

E-mail: sisman@metu.edu.tr and btekin@metu.edu.tr

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Abstract

A systematic construction of the Green's matrix for a second-order self-adjoint matrix differential operator from the linearly independent solutions of the corresponding homogeneous differential equation set is carried out. We follow the general approach of extracting the Green's matrix from the Green's matrix of the corresponding first-order system. This construction is required in the cases where the differential equation set cannot be turned to an algebraic equation set via transform techniques.

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1. Introduction

In physics, matrix differential operators acting on vector functions appear in many different contexts from classical electromagnetism to quantum field theory. The Green's matrices of these operators are needed because of their own physical interpretation as propagators in quantum field theory, or in order to find the solutions of the corresponding non-homogeneous differential equation set.

In most cases, Green's matrices are obtained by using the Fourier transform technique or by using eigenfunction expansions which turn the differential equation set to an algebraic one. However, these techniques are not applicable in some circumstances such as for the matrix differential operator appearing in the (1+1)-dimensional Abelian–Higgs model [1, 2]. In this model, when small field fluctuations around classical field configurations are investigated, the Lagrangian of the theory, which is second order in fluctuations, it involves a 4×4 matrix differential operator. The diagonal entries of this operator are the differential operators of the modified Bessel type. Off-diagonal 'potential' terms are the functions of classical field configurations which are only available as discrete numeric data for the generic values of parameters in the theory. The Green's matrix of this operator is required in calculating the functional determinant of the operator which gives one-loop corrections about a classical solution such as an instanton (for a nice account of functional determinants using the Gel'fand–Yaglom technique see [3]). The Green's function technique is one of the methods used in such a determinant calculation. Due to the existence of the modified Bessel-type operators and the discrete numerical data, it is not possible to apply Fourier transform and eigenfunction expansion in this case. However, in a numerical study, it is relatively easy to obtain linearly independent solutions of the corresponding homogeneous differential equations. Therefore, the construction of the Green's matrix from these solutions as in the case of a single differential operator is required.

It is worth considering the underlying physical problem in some detail in order to see why one encounters a matrix differential operator which is hard to handle. In [4], 't Hooft studied the one-loop tunnelling amplitude in the background of a Yang–Mills instanton for a theory which contains *massless* scalar and fermion fields. In this calculation, field fluctuations do not couple, and the functional determinants of *single* differential operators are calculated. Dunne *et al* [5, 6] extended the instanton determinant calculation to the arbitrary quark mass case. 't Hooft [4] pointed out that in order to remove the infrared divergence of the theory, one needs to introduce the Higgs field. However, due to simple scaling arguments, there is no instanton solution in this case. It is still viable to perform the calculations in the same instanton background, but with certain instanton size, since it was shown in [4] and in [7], with a more elaborate discussion, that the Higgs particle can be taken as approximately massless. However, if one studies the effect of the quantum fluctuations around an instanton background with non-trivial Higgs field configuration, as in the case of the 1+1 Abelian–Higgs model, field fluctuations *do* couple to each other and one needs to struggle with the functional determinant calculation of a matrix differential operator.

In this paper, we consider a generic second-order self-adjoint[1] $n \times n$ matrix differential operator of the form

$$\mathbf{M}_x = \begin{bmatrix} M_{11,x} & V_{12}(x) & \cdots & V_{1n}(x) \\ V_{12}(x) & M_{22,x} & \cdots & V_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ V_{1n}(x) & V_{2n}(x) & \cdots & M_{nn,x} \end{bmatrix}, \tag{1}$$

where the diagonal entries are of the form

$$M_{ii,x} \equiv \left[\frac{\mathrm{d}}{\mathrm{d}x} \left(p_i(x) \frac{\mathrm{d}}{\mathrm{d}x} \right) + q_i(x) \right],$$

and the off-diagonal entries $V_{ij} = V_{ji}$ are just continuous functions. We aim to carry out an explicit construction of the Green's matrix from the linearly independent solutions of the corresponding homogeneous differential equations.

First of all, let us review the literature on Green's matrices. Among the standard textbooks, only the book of Courant and Hilbert [8] involves a short discussion on the properties of the Green's matrix ('tensor' as called in [8]) without a construction. There are also general works on Green's matrices. In [9], Cole studied the Green's matrix of a first-order differential equation set, and constructed the Green's matrix from the solutions of the corresponding homogeneous first-order differential equation set. Cole also constructed the Green's function of a higher derivative differential operator by converting the operator to a first-order matrix differential operator. Reid [10] constructed the Green's matrix for, again, a first-order differential equation set from the solutions of the corresponding homogeneous differential equation set. He also

[1] Self-adjointness requires symmetry of the matrix operator besides the self-adjointness of the differential operators on the diagonals.

developed the properties of the Green's matrices of higher order differential equation sets without giving the Green's matrix. Both Cole and Reid focused on general differential operators which are not self-adjoint. Naimark [11] studied Green's matrices of the nth-order linear matrix differential operators for general homogeneous boundary conditions relating vector function and its derivatives up to the $(n-1)$st order at the boundaries. He gave the Green's matrix constructed from the solutions of homogeneous differential equations for this general system and outlines a way to prove his result by the method of variation of parameters. This result is also for general matrix differential operators which are not self-adjoint. In order to extract the Green's matrix for the second-order self-adjoint case, one needs some of the intermediate results of our construction. The results and analysis of Naimark are too general to be directly applicable to the second-order self-adjoint case that appears frequently in physical applications.

There are also studies which specialize to second-order differential equation sets. Bhagat [12, 13] worked on the case of a second-order self-adjoint 2×2 matrix differential operator. Heimes [14] worked on second-order linear matrix differential operators which are not necessarily self-adjoint. He gave an analogue of our result for second-order linear systems with initial conditions, and without an explicit construction only mentioned that the proper method of obtaining the Green's matrix is to transform the second-order system to the first-order system. Jodar [15] worked on an algebraic construction which may not work in every case.

Applications of Green's matrices to physical problems have been studied in the literature, and authors constructed Green's matrices for their specific problem without a general construction. Baacke [16] gave a construction for a specific matrix differential operator in a heuristic way. His operator involves diagonals of the same self-adjoint operator. He studied one-loop effects in various field theories using Green's matrices found by this construction [1, 2, 16, 17]. In [18], again there is a construction for a specific operator of a 2×2 matrix with the same self-adjoint diagonals. In [18], the main emphasis is on the boundary conditions of the underlying physical system which is a magnetic multilayer structure. Thus, in the literature the Green's matrix of the operator given in (1) seems to be lacking. $\mathbf{M}_x$ is a physically relevant differential operator: for example, it appears in the computation of one-loop quantum corrections around the vortices of the Abelian–Higgs model on the hyperbolic plane [19].

Let us describe the outline of the paper. First of all, we develop the properties of the Green's matrix in section 2. Section 3 is devoted to the construction of the Green's matrix. Construction is carried out in two ways. In the first way, the Green's matrix of a second-order system is extracted from the Green's matrix of the corresponding first-order system. We follow the general approach developed in [9]. The basic idea of transforming the second-order system to the first-order system is also applied in the constructions of [14] and [18]. In the second way of construction, we start with a guess on the form of the Green's matrix, which is along the lines of [16].

To fix the notation, let us note that small bold letters, e.g. $\mathbf{y}$, represent vectors, and capital bold letters, e.g. $\mathbf{G}$, represent matrices. x appearing as an index refers to a differential operator such as $\mathbf{M}_x$. Repeated indices on different matrices are to be summed over, unless otherwise stated. Repeated indices on a single matrix refer to a diagonal element. Also worth noting is the nomenclature: to distinguish the Green's *function* of a single differential operator, we choose the name Green's *matrix* for coupled equations.

2. Properties of the Green's matrix

Let us consider a linear second-order coupled differential equation set:

$$\mathbf{M}_x\mathbf{y}(x) = \mathbf{h}(x), \qquad x \in [a, b], \tag{2}$$

where $\mathbf{y}$, $\mathbf{h}$ are n-dimensional vector functions, and $\mathbf{M}_x$ is the $(n \times n)$-dimensional self-adjoint matrix differential operator of the form (1). The Green's matrix, $\mathbf{G}(x, t)$, of the differential operator, $\mathbf{M}_x$, can be defined with the formal solution

$$\mathbf{y}(x) = \int_a^b \mathrm{d}t\, \mathbf{G}(x, t)\mathbf{h}(t), \tag{3}$$

where $\mathbf{G}(x, t)$ is an $n \times n$ matrix.

In this paper, homogeneous boundary conditions are considered:

$$\mathbf{y}\,(a) = \mathbf{o}, \qquad \mathbf{y}\,(b) = \mathbf{o},$$

where $\mathbf{o}$ is the n-dimensional zero vector. These boundary conditions impose the following conditions on the Green's matrix:

$$\mathbf{G}\,(a, t) = \mathbf{O}, \qquad \mathbf{G}\,(b, t) = \mathbf{O},$$

where $\mathbf{O}$ is an $n \times n$ zero matrix.

The formal solution of the differential equation set implies

$$M_{x,ij}G_{jk}(x, t) = \delta_{ik}\delta\,(x - t)\,, \tag{4}$$

where i, j, k indices run from 1 to n. This relation is an equality of distributions. Integrals of these distributions with a test function yield further properties of the Green's matrix. In obtaining these properties, the cases $i = k$ and $i \neq k$ will be investigated separately.

Case $i \neq k$. The kth column of the Green's matrix is a solution of homogeneous differential equations except the kth equation, as implied by

$$M_{x,ij}G_{jk}(x, t) = 0.$$

Each differential equation involves a term of the form

$$M_{x,ii}G_{ik}(x, t) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q_i(x)\right]G_{ik}(x, t)$$

where there is no summation on i. In order to satisfy these $(n - 1)$ homogeneous differential equations, first- and second-order derivatives in the above term should not yield any singularities, since the other terms of the differential equation contain continuous potentials. Thus, elements of the kth column should be continuous and should have continuous first derivatives for any $x \in [a, b]$, except $G_{kk}(x, t)$ which is investigated in the case $i = k$ below.

Case $i = k$. The kth column of the Green's matrix is a solution of the kth homogeneous differential equation for $x \in [a, t) \cup (t, b]$, as implied by

$$M_{x,kj}G_{jk}(x, t) = \delta\,(x - t)\,,$$

where there is no summation on k. This equation contains the term

$$M_{x,kk}G_{kk}(x, t) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p_k(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q_k(x)\right]G_{kk}(x, t).$$

Since

$$M_{x,kj}G_{jk}(x, t) = 0,$$

for $x \neq t$, $G_{kk}(x,t)$ should be continuous and has continuous first derivatives for the points other than $x = t$.

Let us consider the behaviour at $x = t$. Since all elements of the kth column other than $G_{kk}(x,t)$ are continuous at $x = t$, Dirac delta behaviour comes from $G_{kk}(x,t)$. A discontinuity in $G_{kk}(x,t)$ yields a more severe singularity than the Dirac delta upon taking the second derivative. Thus, $G_{kk}(x,t)$ is continuous at $x = t$ and the first derivative of $G_{kk}(x,t)$ has the usual discontinuity

$$\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}x} G_{ii}(x,t)\bigg|_{t-\epsilon}^{t+\epsilon} = \frac{1}{p_i(t)}.$$

As in the case of the Green's function for a single differential operator, the self-adjointness of $\mathbf{M}_x$ and the homogeneous boundary conditions yield a symmetry property for the Green's matrix:

$$(\mathbf{G}^T)_{li}(x,x_2) M_{x,ij} G_{jk}(x,x_1) = (\mathbf{G}^T)_{li}(x,x_2)\delta_{ik}\delta(x-x_1),$$
$$\Rightarrow G_{il}(x,x_2) M_{x,ij} G_{jk}(x,x_1) = G_{kl}(x,x_2)\delta(x-x_1),$$

and

$$(\mathbf{G}^T)_{ki}(x,x_1) M_{x,ij} G_{jl}(x,x_2) = (\mathbf{G}^T)_{ki}(x,x_1)\delta_{il}\delta(x-x_2),$$
$$\Rightarrow G_{ik}(x,x_1) M_{x,ij} G_{jl}(x,x_2) = G_{lk}(x,x_1)\delta(x-x_2).$$

After integrating the above two equations over the interval $[a,b]$ and subtracting them side by side, one obtains

$$\int_a^b [G_{il}(x,x_2) M_{x,ij} G_{jk}(x,x_1) - G_{ik}(x,x_1) M_{x,ij} G_{jl}(x,x_2)]\,\mathrm{d}x = G_{kl}(x_1,x_2) - G_{lk}(x_2,x_1).$$

Calculating the left-hand side.

For $i = j$:

$$\sum_i \int_a^b \left[G_{il}(x,x_2)\frac{\mathrm{d}}{\mathrm{d}x}\left(p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{ik}(x,x_1)\right) - G_{ik}(x,x_1)\frac{\mathrm{d}}{\mathrm{d}x}\left(p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{il}(x,x_2)\right)\right]\mathrm{d}x.$$

Adding and subtracting $p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{il}(x,x_2)\frac{\mathrm{d}}{\mathrm{d}x}G_{ik}(x,x_1)$ yield

$$\sum_i \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(G_{il}(x,x_2)p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{ik}(x,x_1)\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(G_{ik}(x,x_1)p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{il}(x,x_2)\right)\right]\mathrm{d}x.$$

After the integration, one obtains

$$\sum_i \left[\left(G_{il}(x,x_2)p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{ik}(x,x_1)\right) - \left(G_{ik}(x,x_1)p_i(x)\frac{\mathrm{d}}{\mathrm{d}x}G_{il}(x,x_2)\right)\right]_{x=a}^{x=b} = 0,$$

from $G_{jk}(a,x') = 0$, $G_{jk}\left(b,x'\right) = 0$ for any j and k.

For $i \neq j$:

$$\sum_{i,j;i\neq j}\int_a^b [G_{il}(x,x_2) M_{x,ij} G_{jk}(x,x_1) - G_{ik}(x,x_1) M_{x,ij} G_{jl}(x,x_2)]\,\mathrm{d}x,$$

contains terms like

$$\left\{\int_a^b [G_{1l}(x,x_2) M_{x,12} G_{2k}(x,x_1) - G_{1k}(x,x_1) M_{x,12} G_{2l}(x,x_2)]\,\mathrm{d}x\right\}$$
$$+\left\{\int_a^b [G_{2l}(x,x_2) M_{x,21} G_{1k}(x,x_1) - G_{2k}(x,x_1) M_{x,21} G_{1l}(x,x_2)]\,\mathrm{d}x\right\},$$

which vanish since the matrix differential operator $\mathbf{M}_x$ is symmetric.

Thus, one obtains the symmetry property of the Green's matrix:

$$G_{kl}\left(x_1, x_2\right) = G_{lk}\left(x_2, x_1\right).$$

As a result, the Green's matrix of a second-order self-adjoint matrix differential operator satisfies (4) and the homogeneous boundary conditions. Rewriting them together, we have

$$M_{x,ij} G_{jk}(x, x') = \delta_{ik}\delta(x - x'); \qquad G_{jk}(a, x') = 0, \qquad G_{jk}(b, x') = 0.$$

The properties of the Green's matrix developed in this section can be summarized as follows.

- The kth column of the Green's matrix satisfies the homogeneous differential equations except at one point $x = t$ for the equation $i = k$:

$$M_{x,ij} G_{jk}(x, t) = 0, \qquad x \in \begin{cases} [a, t) \cup (t, b], & i = k, \\ [a, b], & i \neq k. \end{cases}$$

- The Green's matrix is continuous at $x = t$.
- The derivative of the Green's matrix at point $x = t$ is continuous for the off-diagonal elements and has a jump of $1/p_i(t)$ for diagonal elements:

$$\lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}x} G_{ij}(x, t)\bigg|_{t-\epsilon}^{t+\epsilon} = \begin{cases} \dfrac{1}{p_i(t)}, & i = j, \\ 0 & i \neq j. \end{cases}$$

- The Green's matrix has the the following symmetry:

$$\mathbf{G}(x, t) = \mathbf{G}^T(t, x).$$

These properties are also given in [8].

3. Construction of the Green's matrix

A standard way of constructing the Green's function for a second-order linear self-adjoint differential operator

$$L_x \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x),$$

defined in $[a, b]$ is to use the two linearly independent solutions of homogeneous differential equation satisfying

$$L_x u(x) = 0, \qquad u(a) = 0,$$
$$L_x v(x) = 0, \qquad v(b) = 0.$$

The motivation for such a construction follows the observation of two points. First, the Green's function satisfies the homogeneous differential equation, except at $x = t$. The second point is the correspondence between the derivative property of the Green's function:

$$\lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}x} G(x, t)\bigg|_{t-\epsilon}^{t+\epsilon} = \frac{1}{p(t)},$$

and the Wronskian of the solutions u and v:

$$W(u, v) = uv' - vu' = \frac{\text{constant}}{p}. \tag{5}$$

Since the columns of our Green's matrix satisfy the homogeneous differential equation set except at one point $x = t$, it is suggestive that the Green's matrix can be constructed from the solutions of the homogeneous differential equation. In this section, this construction will be given. In section 3.1, analogues of the Wronskian, (5), are obtained. A direct approach for constructing the Green's matrix involves first transforming the second-order differential equation set to a first-order differential equation set. Then, the Green's matrix of the second-order set is extracted from the Green's matrix of the first-order set. This approach is addressed in section 3.2.

3.1. Analogues of the Wronskian

In general, $2n$ linearly independent solutions of

$$\mathbf{M}_x\mathbf{y}(x) = \mathbf{o}$$

can be (re)defined in such a way that n of them satisfy the left boundary condition, and the others satisfy the right boundary condition. Let us call them $\mathbf{u}^\alpha$ and $\mathbf{v}^\beta$, respectively, satisfying

$$\mathbf{u}^\alpha(a) = \mathbf{o}, \qquad \mathbf{v}^\beta(b) = \mathbf{o},$$

where the Greek superscripts label the solutions.

Following the similar steps leading to (5), it is possible to obtain analogue relations for the matrix differential operator case. One can write

$$u_i^\alpha(x)M_{x,ij}v_j^\beta(x) = 0,$$
$$v_i^\beta(x)M_{x,ij}u_j^\alpha(x) = 0.$$

Subtracting side by side yields (the superscripts α and β are suppressed since the equation holds for every α and β; also, the x dependence of the solutions is not explicitly shown up until the final result)

$$u_iM_{x,ij}v_j - v_iM_{x,ij}u_j = 0.$$

Since the matrix differential operator is symmetric, terms like $u_1M_{12}v_2$ and $v_2M_{21}u_1$ cancel each other. After these cancellations, one obtains

$$\begin{aligned}\sum_i(u_iM_{x,ii}v_i - v_iM_{x,ii}u_i) = 0 &\Rightarrow \sum_i[u_i(p_iv_i')' - v_i(p_iu_i')'] = 0,\\ &\Rightarrow \sum_i[(u_ip_iv_i')' - (v_ip_iu_i')'] = 0,\\ &\Rightarrow \sum_i(u_ip_iv_i' - v_ip_iu_i') = \text{constant}.\end{aligned}$$

After putting the superscripts which label the solutions, and showing the explicit x dependence of solutions, one ends up with

$$\sum_i p_i(x)\left(u_i^\alpha(x)\frac{\mathrm{d}}{\mathrm{d}x}v_i^\beta(x) - v_i^\beta(x)\frac{\mathrm{d}}{\mathrm{d}x}u_i^\alpha(x)\right) = C^{\alpha\beta}, \tag{6}$$

where $C^{\alpha\beta}$ are constants. The matrix form of this equation is

$$\mathbf{U}^T(x)\mathbf{P}(x)\mathbf{V}'(x) - (\mathbf{U}')^T(x)\mathbf{P}(x)\mathbf{V}(x) = \mathbf{C}, \tag{7}$$

where $\mathbf{U}$ and $\mathbf{V}$ are $n \times n$ matrices whose columns are the $\mathbf{u}^\alpha$ and $\mathbf{v}^\beta$ vectors, respectively; the $\mathbf{P}$ matrix is defined as

$$\mathbf{P}(x) \equiv \mathrm{diag}\,[p_1(x), p_2(x), \ldots, p_n(x)]\,.$$

Other two relations that can be derived similarly are

$$\mathbf{U}^T(x)\mathbf{P}(x)\mathbf{U}'(x) - (\mathbf{U}')^T(x)\mathbf{P}(x)\mathbf{U}(x) = \mathbf{O}, \tag{8}$$

$$\mathbf{V}^T(x)\mathbf{P}(x)\mathbf{V}'(x) - (\mathbf{V}')^T(x)\mathbf{P}(x)\mathbf{V}(x) = \mathbf{O}. \tag{9}$$

Here, using the boundary conditions $\mathbf{u}^\alpha(a) = \mathbf{o}$ and $\mathbf{v}^\beta(b) = \mathbf{o}$ in (8) and (9), respectively, one finds that the constant matrices on the right-hand sides are equal to zero. Rearranging these equations yields symmetric matrices

$$\mathbf{PU}'\mathbf{U}^{-1} = (\mathbf{PU}'\mathbf{U}^{-1})^T, \tag{10}$$

$$\mathbf{PV}'\mathbf{V}^{-1} = (\mathbf{PV}'\mathbf{V}^{-1})^T, \tag{11}$$

in (a, b), since the $\mathbf{P}$ matrix is diagonal. Using these symmetric forms in (7) yields

$$\mathbf{PV}'\mathbf{V}^{-1} - \mathbf{PU}'\mathbf{U}^{-1} = \left(\mathbf{U}^T\right)^{-1}\mathbf{CV}^{-1}, \tag{12}$$

in (a, b). Note that the right-hand side is necessarily a symmetric matrix due to the symmetry of the left-hand side.

3.2. Green's matrix of the second-order operator from the Green's matrix of the corresponding first-order differential equation set

Let us rewrite the second-order system given in (2) more explicitly

$$\begin{array}{ccccccccc} M_{11,x}y_1(x) & + & V_{12}(x)y_2(x) & + & \cdots & + & V_{1n}(x)y_n(x) & = & h_1(x), \\ V_{12}(x)y_1(x) & + & M_{22,x}y_2(x) & + & \cdots & + & V_{2n}(x)y_n(x) & = & h_2(x), \\ \vdots & + & \vdots & + & \vdots & + & \vdots & = & \vdots \\ V_{1n}(x)y_1(x) & + & V_{2n}(x)y_2(x) & + & \cdots & + & M_{nn,x}y_n(x) & = & h_n(x). \end{array}$$

with the boundary conditions

$$y_i(a) = 0, \qquad y_i(b) = 0.$$

Let us rewrite the system of second-order differential equations as a system of first-order equations with the definitions

$$z_i(x) \equiv y_i(x), \qquad z_{n+i}(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}y_i(x).$$

Then, the first-order system can be written in the following form:

$$\mathbf{z}'(x) = \mathbf{A}(x)\mathbf{z}(x) + \mathbf{f}(x) \tag{13}$$

where

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ -\frac{q_1}{p_1} & -\frac{V_{12}}{p_1} & \cdots & -\frac{V_{1n}}{p_1} & -\frac{p_1'}{p_1} & 0 & \cdots & 0 \\ -\frac{V_{21}}{p_2} & -\frac{q_2}{p_2} & \cdots & -\frac{V_{2n}}{p_2} & 0 & -\frac{p_2'}{p_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{V_{n1}}{p_n} & -\frac{V_{n2}}{p_n} & \cdots & -\frac{q_n}{p_n} & 0 & 0 & \cdots & -\frac{p_n'}{p_n} \end{bmatrix}, \quad \mathbf{z} \equiv \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ z_{n+1} \\ z_{n+2} \\ \vdots \\ z_{2n} \end{bmatrix}, \quad \mathbf{f} \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{h_1}{p_1} \\ \frac{h_2}{p_2} \\ \vdots \\ \frac{h_n}{p_n} \end{bmatrix}.$$

The boundary conditions can be restated in this notation

$$\mathbf{B}_a\mathbf{z}(a) + \mathbf{B}_b\mathbf{z}(b) = \mathbf{o}, \tag{14}$$

where $\mathbf{B}_a$ and $\mathbf{B}_b$ are $2n \times 2n$ matrices in the following form:

$$\mathbf{B}_a = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \qquad \mathbf{B}_b = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{I} & \mathbf{O} \end{bmatrix}, \tag{15}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

The Green's matrix of the first-order system can be defined with the formal solution

$$\mathbf{z}(x) = \int_a^b \mathrm{d}t\mathbf{G}_1(x,t)\mathbf{f}(t).$$

Then, the formal solution of the second-order system is

$$\mathbf{y}(x) = \mathbf{U}_{n\times 2n}\mathbf{z}(x) = \int_a^b \mathrm{d}t\mathbf{U}_{n\times 2n}\mathbf{G}_1(x,t)\mathbf{f}(t),$$

where

$$\mathbf{U}_{n\times 2n} \equiv [\mathbf{I} \quad \mathbf{O}].$$

$\mathbf{f}(t)$ can be rewritten as

$$\mathbf{f}(t) = \mathbf{L}_{2n\times n}\mathbf{P}^{-1}(t)\mathbf{h}(t), \qquad \mathbf{L}_{2n\times n} \equiv \begin{bmatrix}\mathbf{O}\\ \mathbf{I}\end{bmatrix}.$$

Then,

$$\mathbf{y}(x) = \int_a^b \mathrm{d}t\mathbf{U}_{n\times 2n}\mathbf{G}_1(x,t)\mathbf{L}_{2n\times n}\mathbf{P}^{-1}(t)\mathbf{h}(t).$$

Comparing this result with (3) yields

$$\mathbf{G}(x,t) = \mathbf{U}_{n\times 2n}\mathbf{G}_1(x,t)\mathbf{L}_{2n\times n}\mathbf{P}^{-1}(t). \tag{16}$$

Multiplications with $\mathbf{U}_{n\times 2n}$ and $\mathbf{L}_{2n\times n}$ choose an up-right $n \times n$ block of the Green's matrix of the first-order system.

The relation between the Green's matrix of the first-order system and the Green's matrix of the second-order system is established. Let us continue with reproducing the result of Cole [9] for the Green's matrix of a first-order system.

3.2.1. The Green's matrix of a first-order differential equation set. Let us have a 'generic' first-order differential equation set in the form

$$\mathbf{z}'(x) = \mathbf{A}(x)\mathbf{z}(x) + \mathbf{f}(x), \tag{17}$$

where $\mathbf{z}$, $\mathbf{f}$ are l the dimensional vectors and $\mathbf{A}$ is an $(l \times l)$-dimensional matrix.

First, assume a particular solution of the form

$$\mathbf{z}_p(x) = \mathbf{W}(x)\mathbf{g}(x)$$

where $\mathbf{g}$ is an unknown column vector, and the so-called fundamental matrix $\mathbf{W}$ is the matrix whose columns are the l linearly independent solutions of the homogeneous differential equation set, $\mathbf{z}'(x) = \mathbf{A}(x)\mathbf{z}(x)$; i.e.

$$\mathbf{W}'(x) = \mathbf{A}(x)\mathbf{W}(x).$$

Then, guessing the particular solution in the non-homogeneous equation yields the formal solution for $\mathbf{g}(x)$ as

$$\mathbf{g}(x) = \int_a^x \mathrm{d}t\, \mathbf{W}^{-1}(t)\mathbf{f}(t).$$

Therefore, the general solution for the generic first-order differential equation set is

$$\mathbf{z}(x) = \mathbf{W}(x)\int_a^x \mathrm{d}t\, \mathbf{W}^{-1}(t)\mathbf{f}(t) + \mathbf{W}(x)\mathbf{c}, \tag{18}$$

where $\mathbf{c}$ is a constant vector.

Any well-posed boundary condition which yields a unique solution to the boundary value problem can be put in a matrix form. Consider the boundary conditions

$$\mathbf{B}_a \mathbf{z}\,(a) + \mathbf{B}_b \mathbf{z}\,(b) = \mathbf{o}. \tag{19}$$

Applying these boundary conditions to the general solution fixes $\mathbf{c}$ as

$$\mathbf{c} = -\mathbf{D}^{-1}\mathbf{B}_b \mathbf{W}\,(b) \int_a^b \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t)$$

where $\mathbf{D}$ is defined by

$$\mathbf{D} \equiv \mathbf{B}_a \mathbf{W}\,(a) + \mathbf{B}_b \mathbf{W}\,(b)\,.$$

Using this result in the general solution, one obtained

$$\mathbf{z}(x) = \mathbf{W}(x) \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t) - \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_b \mathbf{W}\,(b) \int_a^b \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t). \tag{20}$$

This result can be written in a final form by rearranging the first term on the right-hand side as

$$\begin{aligned}
\mathbf{W}(x) \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t) &= \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{D} \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t),\\
&= \mathbf{W}(x)\mathbf{D}^{-1}\left\{\mathbf{B}_a \mathbf{W}\,(a) + \mathbf{B}_b \mathbf{W}\,(b)\right\} \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t),\\
&= \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_a \mathbf{W}\,(a) \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t)\\
&\quad + \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_b \mathbf{W}\,(b) \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t).
\end{aligned}$$

Using this in (20) yields

$$\mathbf{z}(x) = \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_a \mathbf{W}\,(a) \int_a^x \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t) - \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_b \mathbf{W}\,(b) \int_x^b \mathrm{d}t\,\mathbf{W}^{-1}(t)\mathbf{f}(t),$$

or

$$\mathbf{z}(x) = \int_a^b \mathrm{d}t\;\mathbf{G}_1(x,t)\mathbf{f}(t)$$

where the Green's matrix is given as

$$\mathbf{G}_1(x,t) \equiv \begin{cases} -\mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_b \mathbf{W}\,(b)\,\mathbf{W}^{-1}(t), & x < t,\\ \mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_a \mathbf{W}\,(a)\,\mathbf{W}^{-1}(t), & x > t. \end{cases} \tag{21}$$

3.2.2. Green's matrix of the second-order system. Using (16), the Green's matrix for the second-order system is

$$\mathbf{G}(x,t) = \begin{cases} -\mathbf{U}_{n\times 2n}\mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_b \mathbf{W}\,(b)\,\mathbf{W}^{-1}(t)\mathbf{L}_{2n\times n}\mathbf{P}^{-1}(t), & x \leqslant t,\\ \mathbf{U}_{n\times 2n}\mathbf{W}(x)\mathbf{D}^{-1}\mathbf{B}_a \mathbf{W}\,(a)\,\mathbf{W}^{-1}(t)\mathbf{L}_{2n\times n}\mathbf{P}^{-1}(t), & x \geqslant t. \end{cases} \tag{22}$$

Note that $\mathbf{W}$ which is the fundamental matrix of the first-order set is the Wronskian matrix of the second order set. With this result, we actually achieve our goal which is to construct the Green's matrix of the second order self-adjoint matrix differential operator from the linearly independent solutions of the corresponding homogenous differential equation set. However, we continue to work on this result in order to find a more compact form and to find forms in which the properties of the Green's matrix are more transparent.

First of all, let us transform the Green's matrix to a form in which boundary conditions are explicit. In order to achieve this goal, let us assume that the $2n$ linearly independent solutions forming the Wronskian matrix $\mathbf{W}$ is chosen in such a way that n of them satisfy one boundary condition, and the other n satisfy the other boundary condition; i.e.

$$\mathbf{M}_x\mathbf{u}^{\alpha}(x) = \mathbf{o}, \qquad \mathbf{u}^{\alpha}(a) = \mathbf{o},$$
$$\mathbf{M}_x\mathbf{v}^{\beta}(x) = \mathbf{o}, \qquad \mathbf{v}^{\beta}(b) = \mathbf{o},$$

where $\alpha, \beta = 1, \ldots, n$. Then, let us choose $\mathbf{W}$ in the form

$$\mathbf{W} = \begin{bmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{U}' & \mathbf{V}' \end{bmatrix},$$

where $\mathbf{U}$ and $\mathbf{V}$ are the $n \times n$ matrices whose columns are $\mathbf{u}^{\alpha}$ and $\mathbf{v}^{\beta}$, respectively. Using this form of $\mathbf{W}$ greatly simplifies the matrix multiplications given in (22) and yields the results

$$\mathbf{D}^{-1}\mathbf{B}_b\mathbf{W}(b) = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \qquad \mathbf{D}^{-1}\mathbf{B}_a\mathbf{W}(a) = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}.$$

The block inverse of the Wronskian matrix can be given as (see the appendix)

$$\mathbf{W}^{-1} = \begin{bmatrix} \mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{V}(\mathbf{V}' - \mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{U}'\mathbf{U}^{-1} & -\mathbf{U}^{-1}\mathbf{V}(\mathbf{V}' - \mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1} \\ -(\mathbf{V}' - \mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{U}'\mathbf{U}^{-1} & (\mathbf{V}' - \mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1} \end{bmatrix},$$

where each matrix is a function of t. Note that $\mathbf{U}(t)$ is singular at $x = t = a$. Thus, $x \leqslant t$ part of the Green's matrix should be written in such a way that the $x = a$ boundary value is given separately.

Putting these results in (22) yields a form where boundary conditions are explicitly satisfied:

$$\mathbf{G}(x,t) = \begin{cases} \mathbf{O}, & a = x \leqslant t, \\ \mathbf{U}(x)\mathbf{U}^{-1}(t)\mathbf{V}(t)(\mathbf{V}'(t) - \mathbf{U}'(t)\mathbf{U}^{-1}(t)\mathbf{V}(t))^{-1}\mathbf{P}^{-1}(t), & a < x \leqslant t, \\ \mathbf{V}(x)(\mathbf{V}'(t) - \mathbf{U}'(t)\mathbf{U}^{-1}(t)\mathbf{V}(t))^{-1}\mathbf{P}^{-1}(t), & x \geqslant t. \end{cases} \tag{23}$$

Or, after rearranging, one has a more symmetric form:

$$\mathbf{G}(x,t) = \begin{cases} \mathbf{O} & a = x \leqslant t, \\ \mathbf{U}(x)\mathbf{U}^{-1}(t)[\mathbf{P}(t)(\mathbf{V}'(t)\mathbf{V}^{-1}(t) - \mathbf{U}'(t)\mathbf{U}^{-1}(t))]^{-1}, & a < x \leqslant t, \\ \mathbf{V}(x)\mathbf{V}^{-1}(t)[\mathbf{P}(t)(\mathbf{V}'(t)\mathbf{V}^{-1}(t) - \mathbf{U}'(t)\mathbf{U}^{-1}(t))]^{-1}, & b > x \geqslant t, \\ \mathbf{O} & b = x \geqslant t. \end{cases} \tag{24}$$

With the help of (12), a final compact form of the Green's matrix is obtained as

$$\mathbf{G}(x,t) = \begin{cases} \mathbf{U}(x)(\mathbf{C}^T)^{-1}\mathbf{V}^T(t), & x \leqslant t, \\ \mathbf{V}(x)\mathbf{C}^{-1}\mathbf{U}^T(t), & x \geqslant t. \end{cases} \tag{25}$$

Or, writing in terms of the elements

$$G_{ij}(x,t) = \left(\mathbf{C}^{-1}\right)_{\beta\alpha} \begin{cases} u_i^{\alpha}(x)v_j^{\beta}(t), & x \leqslant t, \\ v_i^{\beta}(x)u_j^{\alpha}(t), & x \geqslant t. \end{cases} \tag{26}$$

where there is summation on the Greek indices.

In all of the above forms of the Green's matrix, some properties are explicit, while the others are not. Now, let us investigate these properties.

3.2.3. Verifying the properties of the Green's matrix. Let us show that the Green's matrix that we have constructed satisfies the properties listed in section 2.

- It obviously satisfies the homogeneous boundary conditions in the forms starting with (23).

- Its columns are formed by the solutions of the homogeneous differential equation.
- It is continuous at $x = t$, which is explicit in (23) and (24).
- In the forms (25) and (26), the symmetry property is explicit. Let us write $\mathbf{G}^T\,(t, x)$:

$$\mathbf{G}^T\,(t, x) = \begin{cases} \mathbf{V}(x)\mathbf{C}^{-1}\mathbf{U}^T(t), & t \leqslant x, \\ \mathbf{U}(x)(\mathbf{C}^{-1})^T\mathbf{V}^T(t), & t \geqslant x, \end{cases}$$

  which is simply equal to $\mathbf{G}(x, t)$.
- The derivative property of the Green's matrix can be given in the matrix form as

$$\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{G}(x, t)\bigg|_{t-\epsilon}^{t+\epsilon} = \mathbf{P}^{-1}(t). \tag{27}$$

  Using the Green's matrix form given in (24):

$$\mathbf{V}'(t)\mathbf{V}^{-1}(t)[\mathbf{P}(t)(\mathbf{V}'(t)\mathbf{V}^{-1}(t) - \mathbf{U}'(t)\mathbf{U}^{-1}(t))]^{-1} \\ - \mathbf{U}'(t)\mathbf{U}^{-1}(t)[\mathbf{P}(t)(\mathbf{V}'(t)\mathbf{V}^{-1}(t) - \mathbf{U}'(t)\mathbf{U}^{-1}(t))]^{-1} = \mathbf{P}^{-1}(t).$$

3.3. Construction with a guess

Now, let us try to construct the Green's matrix in a similar way as in the case of a single differential operator. A somewhat similar derivation was given in [16] for a specific case. Since the columns of the Green's matrix are the solutions of the homogeneous differential equation, the Green's matrix should have the following form in order to satisfy the boundary conditions:

$$\mathbf{G}(x, t) = \begin{cases} \mathbf{U}(x)\mathbf{S}(t), & x < t, \\ \mathbf{V}(x)\mathbf{T}(t), & x > t, \end{cases}$$

where $\mathbf{S}(t)$ and $\mathbf{T}(t)$ are the unknown matrices. In this form, each column is a linear combination of the linearly independent solutions. Surely, this form is the most general guess satisfying the boundary conditions and the homogeneous differential equation set. After putting the proposal in the derivative property in (27), one obtains

$$\mathbf{V}'(t)\mathbf{T}(t) - \mathbf{U}'(t)\mathbf{S}(t) = \mathbf{P}^{-1}(t).$$

Using (12), one has

$$\mathbf{P}^{-1} = (\mathbf{V}'\mathbf{V}^{-1} - \mathbf{U}'\mathbf{U}^{-1})\mathbf{V}\mathbf{C}^{-1}\mathbf{U}^T.$$

Putting it in the derivative property, and using the symmetry of $\mathbf{V}\mathbf{C}^{-1}\mathbf{U}^T$ yield

$$\mathbf{V}'\mathbf{T} - \mathbf{U}'\mathbf{S} = \mathbf{V}'\mathbf{C}^{-1}\mathbf{U}^T - \mathbf{U}'(\mathbf{C}^{-1})^T\mathbf{V}^T.$$

Then, the unknown $\mathbf{S}$ and $\mathbf{T}$ matrices are

$$\mathbf{S} = (\mathbf{C}^{-1})^T\mathbf{V}^T, \\ \mathbf{T} = \mathbf{C}^{-1}\mathbf{U}^T.$$

This result yields the same form given in (25).

4. Discussion and conclusion

In this paper, we constructed the Green's matrix of a second-order self-adjoint matrix differential operator. This construction is useful especially in the numerical studies, since obtaining the linearly independent solutions of the corresponding homogeneous differential equation set is easy with linearly independent initial conditions. Just after obtaining the set of linearly independent solutions, one may directly use (22) and obtain the Green's matrix without considering the boundary behaviour of the solutions. However, once the linearly independent solutions are redefined such that half of them satisfy one homogeneous boundary condition and the other half the other boundary, or are obtained directly in this way with proper boundary conditions, the compact form of the Green's matrix in (25) can be used by calculating the constant matrix in (7). By taking this route, one may avoid numerical errors coming from correspondingly greater amount of matrix multiplications and inversions. Also, in the cases where solving the corresponding homogeneous differential equation set analytically is easy, construction of the Green's matrix by use of (25) may become as easy as other techniques.

Extracting the Green's matrix of a higher order differential equation set from the corresponding first-order differential equation set is a useful technique which is not well known in physics literature. Although we construct the Green's matrix of a second-order self-adjoint matrix differential operator by using the Green's matrix of the corresponding first-order differential equation set, due to its physical relevance, Green's matrix for any higher order linear (matrix) operator, either having self-adjointness property or not, can be extracted from the Green's matrix of the corresponding first-order differential equation set.

A final comment on the boundary conditions is that a differential equation set satisfying boundary conditions other than the homogeneous ones can be handled as in the case of the single differential operator. Construction of a Green's matrix satisfying boundary conditions other than homogeneous ones can be handled by the first method given in this paper.

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Appendix. Inverse of a block matrix

We reproduce the derivation given by Thornburg [20]. Let us have a $2n \times 2n$ matrix in the form

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the $n \times n$ matrices. The inverse of this matrix can be determined by obtaining the block decomposition of this matrix.

Inverse of the following matrix forms can be found easily:

$$\begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}^{-1} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{O} & \mathbf{B} \\ \mathbf{C} & \mathbf{O} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{O} & \mathbf{C}^{-1} \\ \mathbf{B}^{-1} & \mathbf{O} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{B} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{C} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{C} & \mathbf{I} \end{bmatrix}.$$

If we block decompose of a general matrix in such a way that the above forms appear, then taking inverse can be handled by using the above relations. In order to obtain the block decomposition, one can use the following equation:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ \mathbf{H} \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} \mathbf{AE} + \mathbf{BF} = \mathbf{G}, \\ \mathbf{CE} + \mathbf{DF} = \mathbf{H}. \end{array}$$

This equation set can be solved for $\mathbf{F}$ by multiplying the first equation with $-\mathbf{CA}^{-1}$ and summing with the second one. These operations are equal to multiplying coefficient matrix with

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{CA}^{-1} & \mathbf{I} \end{bmatrix}$$

from left. Then, coefficient matrix becomes

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{O} & \mathbf{D}-\mathbf{CA}^{-1}\mathbf{B} \end{bmatrix}$$

where $\mathbf{S}_A \equiv \mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}$ is called the Schur complement of $\mathbf{A}$. In order to solve equations in $\mathbf{E}$, the second equation should be multiplied with $-\mathbf{BS}_A^{-1}$ and summed with the first equation. These operations are equal to multiplying the modified coefficient matrix with

$$\begin{bmatrix} \mathbf{I} & -\mathbf{BS}_A^{-1} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}$$

from left. Afterwards, the coefficient matrix becomes

$$\begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_A \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} \mathbf{I} & -\mathbf{BS}_A^{-1} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{CA}^{-1} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_A \end{bmatrix}$$

which yields the block decomposed form

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{CA}^{-1} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{BS}_A^{-1} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_A \end{bmatrix}.$$

Then, the inverse of the $2n \times 2n$ matrix can be found as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{BS}_A^{-1}\mathbf{CA}^{-1} & -\mathbf{A}^{-1}\mathbf{BS}_A^{-1} \\ -\mathbf{S}_A^{-1}\mathbf{CA}^{-1} & \mathbf{S}_A^{-1} \end{bmatrix}.$$

Using the above result, the inverse of the Wronskian matrix is

$$\begin{bmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{U}' & \mathbf{V}' \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{U}^{-1}+\mathbf{U}^{-1}\mathbf{V}(\mathbf{V}'-\mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{U}'\mathbf{U}^{-1} & -\mathbf{U}^{-1}\mathbf{V}(\mathbf{V}'-\mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1} \\ -(\mathbf{V}'-\mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{U}'\mathbf{U}^{-1} & (\mathbf{V}'-\mathbf{U}'\mathbf{U}^{-1}\mathbf{V})^{-1} \end{bmatrix}$$

Note that the linear independence of the solutions of the corresponding homogeneous differential equation set implies invertibility of the Wronskian matrix and its four elements.

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